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Logarithmic potential β-ensembles and Feynman graphs

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To my teacher, Andrei Alexeevich Slavnov, who showed me the beauty of Feynman graphs

We present the diagrammatic technique for calculating the free energy of the matrix eigenvalue model (the model with arbitrary power β by the Vandermonde) to all orders of $1/N$ expansion in the case where the limiting eigenvalue distribution spans arbitrary (but fixed) number of disjoint intervals (curves) and when logarithmic terms are present. This diagrammatic technique is corrected and refined as compared to our first paper with B.Eynard of year 2006.

1 Introduction

The standard Hermitian one-matrix model is determined, after integrating out angular degrees of freedom, by the N-fold integral over eigenvalues of the form

$$\int \prod_{i=1}^{N} dx_i \, \Delta(x)^2 e^{-\frac{1}{\hbar}\sum_{i=1}^{N} V(x_i)}, \quad \hbar = t_0/N$$

where $\Delta(x)$ is the Vandermonde determinant of eigenvalues x_i, $\hbar$ is the formal expansion parameter, and t_0 is the normalized number of eigenvalues. The β-ensembles, or β-eigenvalue models, are generalizations of this integral obtained by replacing the exponent of the Vandermonde determinant by an arbitrary positive number 2β. In the proper normalization, we then evaluate the (formal) integral

$$\int \prod_{i=1}^{N} dx_i \, |\Delta(x)|^{2\beta} e^{-\frac{\sqrt{\beta}}{\hbar}\sum_{i=1}^{N} V(x_i)} = e^{-\mathcal{F}}. \tag{1}$$

The three values $\beta = 1/2, 1, 2$ correspond in the Wigner classification by the respective ensembles of orthogonal, Hermitian, and self-dual quaternionic matrices, but we can put forward the problem of calculating the perturbation expansion of the β-model free energy $\mathcal{F}$ for arbitrary value of β.

In our joint paper with B. Eynard [5], we developed the perturbation expansion solution to the integral (1). Our procedure enables constructing all the correlation function and the free energy expansion terms order by order of the double asymptotic expansion in the two parameters: $\hbar^2$ and

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$(\sqrt{\beta}-\sqrt{\beta^{-1}})\hbar$. The terms of these expansions (for all the terms except few leading terms) were expressed using the special Feynman-like diagrammatic technique, which is a generalization of the diagrammatic technique originally constructed in our paper [4] for the Hermitian one-matrix model.

The recent interest to the β-eigenvalue models is due to the Alday, Gaiotto and Tachikawa conjecture [1] relating instantonic generating functions to the conformal blocks of the Liouville theory (see also [9]). These conformal blocks were in turn associated with the asymptotic limits of the β-ensembles containing logarithmic terms in the potential. More precisely, it was shown (see [7] and references therein) that the Penner-like model with three logarithmic terms proposed by Dijkgraaf and Vafa [6] correctly reproduces the standard results of $N=2$ supersymmetric gauge theories.

On the other hand, explicit calculations performed by Sara Pasquetti and Marcos Mariño had revealed that applying the original diagrammatic technique of [5] results in a nonsymmetric result for the two-point correlation function already in the subleading expansion order. This clearly indicated the incompleteness of the diagrammatic technique of our first paper.

Therefore, the aim of this paper is twofold: on the one hand, we extend the pattern of the paper [5] including the logarithmic potentials into consideration and, on the other hand, we formulate a new *improved diagrammatic technique* endowed with the new selection rules that excludes part of diagrams that caused a mismatch when calculating correlation functions.

As a nearest perspective, it is interesting to include the hard edges into consideration, i.e., to construct the consistent diagrammatic technique for the β-ensembles in the situation where we put explicit restrictions on the intervals of the eigenvalue distribution; the first step in this direction was done in the recent paper [2] in which a one-cut solution was investigated and results for few lower order terms of the free-energy expansion were derived.

2 Eigenvalue models in $1/N$-expansion

In this section we show how the technique of Feynman graph expansion elaborated in the case of Hermitian one-matrix model [8], [4] can be adapted to solving the (formal) eigenvalue model with the action

$$\int \prod_{i=1}^{N} dx_i\, \Delta^{2\beta} e^{-\frac{\sqrt{\beta}}{\hbar}\sum_{i=1}^{N}\widehat{V}(x_i)} \prod_{k=1}^{t}\left(\prod_{i=1}^{N}|x_i-\alpha_k|^{\frac{\sqrt{\beta}}{\hbar}\gamma_k}\right) = e^{-\mathcal{F}}, \tag{2}$$

where $\widehat{V}(x)=\sum_{s\geq 1} t_s x^s$, $\hbar = t_0/N$ is a formal expansion parameter, and t_0 is the normalized eigenvalue number. The integration in (2) goes over N variables x_i having the sense of eigenvalues of the Hermitian matrices for $\beta=1$, orthogonal matrices for $\beta=1/2$, and symplectic matrices for $\beta=2$. In what follows, we set β to be arbitrary positive number. The integration may go over curves in the complex plane of each of N variables x_i. For $\beta\neq 1$, no topological expansion in even powers of $\hbar$ exists and we rather have the expansion in all integer powers of $\hbar$. We also assume the potential $\widehat{V}(p)$ to be a polynomial of the fixed degree $m+1$, or, if we include logarithmic terms in the potential introducing

$$V(x_i)=\widehat{V}(x_i)+\sum_{k=1}^{t}\gamma_k \log|x_i-\alpha_k|, \tag{3}$$

then it suffices to demand the derivative $V'(p)$ to be a rational function [8].

For brevity, we let the symbol $\operatorname{tr} f(X)$ to denote $\sum_{i=1}^{N} f(x_i)$ for any function $f(x)$.

2.1 The loop equation and resolvents

The averages corresponding to partition function (2) are defined in a standard way:

$$\langle F(X)\rangle = \frac{1}{Z}\int_N DX\,F(X)\,\exp\left(-\frac{\sqrt{\beta}t_0}{\hbar}\mathrm{tr}\,V(X)\right) \tag{4}$$

with the total potential $V(x)$, and we introduce their formal generating functionals: the one-point resolvent

$$W(p) = \hbar\sqrt{\beta}\sum_{k=0}^{\infty}\frac{\langle \mathrm{tr}\,X^k\rangle}{p^{k+1}} \tag{5}$$

as well as the s-point resolvents ($s \geq 2$)

$$W(p_1,\dots,p_s) = (\hbar\sqrt{\beta})^{2-s}\sum_{k_1,\dots,k_s=1}^{\infty}\frac{\langle \mathrm{tr}\,X^{k_1}\cdots\mathrm{tr}\,X^{k_s}\rangle_{\mathrm{conn}}}{p_1^{k_1+1}\cdots p_s^{k_s+1}} = \hbar^{2-s}\left\langle \mathrm{tr}\,\frac{1}{p_1-X}\cdots\mathrm{tr}\,\frac{1}{p_s-X}\right\rangle_{\mathrm{conn}} \tag{6}$$

where the subscript "conn" pertains to the connected part.

These resolvents are obtained from the free energy $\mathcal{F}$ through the action

$$\begin{aligned} W(p_1,\dots,p_s) &= -\hbar^2\frac{\partial}{\partial V(p_s)}\frac{\partial}{\partial V(p_{s-1})}\cdots\frac{\partial\mathcal{F}}{\partial V(p_1)} = \\ &= \frac{\partial}{\partial V(p_s)}\frac{\partial}{\partial V(p_{s-1})}\cdots\frac{\partial}{\partial V(p_2)}W(p_1), \end{aligned} \tag{7}$$

of the *loop insertion operator*

$$\frac{\partial}{\partial V(p)} \equiv -\sum_{j=1}^{\infty}\frac{1}{p^{j+1}}\frac{\partial}{\partial t_j}. \tag{8}$$

Therefore, if one knows exactly the one-point resolvent for arbitrary potential, all multi-point resolvents can be calculated by induction.

The loop insertion operator has an interpretation in the theory of symmetric forms. Let us set into the correspondence to an s-point correlation function $W_{p_1,\dots,p_s}$ the symmetric form $W_{p_1,\dots,p_s}dp_1\cdots dp_s$ that is meromorphic on our Riemann surface. We let $\Omega^s_{1,0}$ denote the (infinite-dimensional) space of such forms. Then, the action of the loop insertion operator increases the order of this form by one,

$$\frac{\partial}{\partial V}\ :\Omega^s_{1,0}\mapsto\Omega^{s+1}_{1,0}. \tag{9}$$

We now consider the perturbation expansion of quantities in (7). In the above normalization, the $\hbar$-expansion has the form

$$W(p_1,\dots,p_s) = \sum_{r=0}^{\infty}\hbar^r W_{r/2}(p_1,\dots,p_s), \quad s\geq 1, \tag{10}$$

and for $\beta\neq 1$ it comprises both even and odd powers of $\hbar$, whereas in the Hermitian matrix model case ($\beta = 1$) the corresponding expansion (the so-called genus expansion) goes over only even powers of $\hbar$.

We obtain the *master loop equation* of the eigenvalue model (2) if we perform the changing $x_i \to x_i + \frac{\epsilon}{x_i - p}$ of the integration variable. The resulting (**exact**) equation first takes the following form in

terms of the means:

$$
\begin{aligned}
&-\sum_{i,j=1}^{N}\left\langle \frac{\beta}{(x_i-p)(x_j-p)}\right\rangle-\sum_{i=1}^{N}\left\langle \frac{1-\beta}{(x_i-p)^2}\right\rangle+\frac{\sqrt{\beta}}{\hbar}\widehat{V}'(p)\sum_{i=1}^{N}\left\langle \frac{1}{x_i-p}\right\rangle\\
&+\sum_{i=1}^{N}\left\langle \sum_{s=1}^{m+1} st_s\frac{x_i^{s-1}-p^{s-1}}{x_i-p}\right\rangle-\sum_{k=1}^{t}\frac{\sqrt{\beta}}{\hbar}\gamma_k\frac{1}{\alpha_k-p}\sum_{i=1}^{N}\left[\left\langle \frac{1}{x_i-\alpha_k}\right\rangle-\left\langle \frac{1}{x_i-p}\right\rangle\right]=0. \qquad (11)
\end{aligned}
$$

We let $P_{m-1}(p)$ denote the degree-$(m-1)$ polynomial $\sum_{i=1}^{N}\left\langle \sum_{s=1}^{m+1} st_s\frac{x_i^{s-1}-p^{s-1}}{x_i-p}\right\rangle$. Note also that the resolvent $W(p)$ must be nonsingular at $p=\alpha_k$ for any k. Then Eq. (11) becomes in normalization (5):

$$
\begin{aligned}
&-W^2(p)-\hbar^2W(p,p)+\hbar\left(\sqrt{\beta}-\sqrt{\beta}^{-1}\right)W'(p)+\widehat{V}'(p)W(p)+P_{m-1}(p)-\\
&\qquad-\sum_{k=1}^{t}\gamma_k\frac{1}{\alpha_k-p}\left[W(\alpha_k)-W(p)\right]=0. \qquad (12)
\end{aligned}
$$

2.2 Solution at large N

Here we specially consider the limit as $\hbar\to 0$ in (12). Note that $W^{(0)}(p)|_{p\to\infty}=t_0/p+O(p^{-2})$.

For $W^{(0)}(p)$ we have the algebraic equation to resolve which it is useful to introduce the function

$$
y(p)=W(p)-\frac{1}{2}\left[\widehat{V}'(p)-\sum_{k=1}^{t}\gamma_k\frac{1}{p-\alpha_k}\right]:=W(p)-\frac{1}{2}V'(p) \qquad (13)
$$

for which Eq. (12) gives the full square:

$$
y(p)=\frac{\sqrt{P_{2m+2t}(p)}}{\prod_{k=1}^{t}(p-\alpha_k)}. \qquad (14)
$$

The function $y(p)$ may have up to $2m+2t$ branching points thus defining a hyperelliptic Riemann surface of maximum genus $m+t-1$. We let $2n$ be the actual number of the distinct branching points μ_r (some branching points may merge producing double points), and the actual genus is then $n-1$. We assume that we have therefore n disjoint intervals A_i of eigenvalue distribution and $n-1$ "forbidden zones" B_i. We treat $n-1$ among n intervals A_i as a-cycles and we then let $\sum_{j=i}^{n-1}B_j$ to be the corresponding b_i-cycle (one half of which goes along the physical sheet and the other half goes along the second, unphysical sheet). So, in general, we assume that $y(p)$ has the form

$$
y(p)=M_{m+t-n}(p)\frac{\sqrt{\prod_{r=1}^{2n}(p-\mu_r)}}{\prod_{k=1}^{t}(p-\alpha_k)},
$$

where the polynomial $M_{m+t-n}(p)$ arises from merging branching points of $P_{2m+2t}(p)$.

The $m+t-n+1$ coefficients of the polynomial $M_{m+t-n}(p)$ and the $2n$ branching points μ_r are to be determined from the following conditions:

- (asymptotic condition at infinity)

$$
\lim_{p\to\infty}y(p)=-\frac{1}{2}\widehat{V}'(p)+\left(\frac{1}{2}\sum_{k=1}^{t}\gamma_k+t_0\right)\frac{1}{p}+O(p^{-2}) \qquad m+2 \text{ conditions;}
$$

- (regularity at $p = \alpha_k$ of $W(p)$):

$$\frac{\sqrt{P_{2m+2t}(\alpha_k)}}{\prod_{l\neq k}(\alpha_k - \alpha_l)} = \frac{1}{2}\gamma_k \qquad t \text{ conditions;}$$

- (equality of chemical potentials on all cuts)

$$\oint_{B_i} ydp = 0 \qquad n-1 \text{ conditions.}$$

Note that, in the context of the Seiberg–Witten theory, the last $n-1$ conditions can acquire other forms, for instance,

- (fixing filling fractions on the intervals of eigenvalue distribution)

$$\oint_{A_i} ydp = \epsilon_i \qquad n-1 \text{ conditions.}$$

So, in total we have exactly the desired number of $m+t+n+1$ conditions.

The recurrent procedure in the subsequent sections is such that all the geometry properties of the theory (Bergmann kernels, etc.) is completely determined by the *reduced* hyperelliptic Riemann surface

$$\tilde{y}^2 = \prod_{r=1}^{2n}(p-\mu_r),$$

where we lose all the information on the rational function $Q(p) := M_{m+t-n}(p)/(\prod_{k=1}^{t}(p-\alpha_k))$.

All the rest of the text is totally insensitive to the nature of the rational function $Q(p)$.

We need to introduce the *Bergman kernel*: the symmetric ($B(p,q) = B(q,p)$) bi-differential that has the form

$$B(p,q) := B(x(p),x(q))dx(p)dx(q) \sim \frac{dx(p)dx(q)}{(x(p)-x(q))^2} \text{ as } x(p)\to x(q), \ \oint_{B_j} B(\cdot,p) = 0$$

in the local coordinate $x(p)$.

We also need its primitive (a one-differential in r):

$$dE_{p,q}(r) := \int_{x(p)}^{x(q)} B(\cdot, r).$$

Note that both $B(p,q)$ and $dE_{p,q}(r)$ depend only on "reduced" Riemann surface (are determined completely by the branching points μ_r). Definitely, they take the simplest form for **one-cut** solutions: if we have just two branching points μ_1 and μ_2, then, for $\bar{q}$ being the image of q on the other sheet,

$$dE_{q,\bar{q}}(r) = \frac{\sqrt{(q-\mu_1)(q-\mu_2)}}{(r-q)\sqrt{(r-\mu_1)(r-\mu_2)}}dr;$$

for higher-genus spectral curves, this expression contains also projections terms proportional to Abelian differentials normalized to B-cycles.

2.3 Higher-order corrections

The β-dependence enters (12) only through the combination

$$\zeta = \sqrt{\beta} - \sqrt{\beta^{-1}}, \tag{15}$$

and, assuming $\beta \sim O(1)$, we have the free energy expansion of the form

$$\mathcal{F} \equiv \mathcal{F}(\hbar, \zeta, t_0, t_1, t_2, \ldots) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \hbar^{2k+l-2} \zeta^l \mathcal{F}_{k,l}. \tag{16}$$

Substituting expansion (10) in Eq. (12), we find that $W_g(p)$ for $g \geq 1/2$ satisfy the equation

$$\begin{aligned} -2y(p)W_g(p) &= \sum_{g'=1/2}^{g-1/2} W_{g'}(p)W_{g-g'}(p) + \frac{\partial}{\partial V(p)} W_{g-1}(p) + \zeta \frac{\partial}{\partial p} W_{g-1/2}(p) \\ &\quad + P_{m-2}^{(g)}(p) + \sum_{k=1}^{t} \gamma_k \frac{1}{p-\alpha_k} W_g(\alpha_k) \end{aligned} \tag{17}$$

In Eq. (17), $W_g(p)$ is expressed through only the $W_{g_i}(p)$ for which $g_i < g$. This fact permits developing the iterative procedure.

In analogy with (16), it is convenient to expand multiresolvents $W_g(\cdot)$ in ζ:

$$W_g(p_1, \ldots, p_s) = \begin{cases} \sum_{l=0}^{g} \zeta^{2l} W_{g-l,\ 2l}(p_1, \ldots, p_s), & g \in \mathbb{Z} \\ \sum_{l=0}^{g-1/2} \zeta^{2l+1} W_{g-l-1/2,\ 2l+1}(p_1, \ldots, p_s), & g \in \mathbb{Z} + 1/2 \end{cases} \tag{18}$$

Then, obviously, (17) becomes

$$\begin{aligned} -2y(p)W_{k,l}(p) &= \sum_{\substack{k_1 \geq 0, l_1 \geq 0 \\ k_1 + l_1 > 0}} W_{k_1,l_1}(p) W_{k-k_1,l-l_1}(p) + \frac{\partial}{\partial V(p)} W_{k-1,l}(p) + \zeta \frac{\partial}{\partial p} W_{k,l-1}(p) \\ &\quad + P_{m-2}^{(k,l)}(p) + \sum_{s=1}^{t} \gamma_s \frac{1}{p-\alpha_s} W_{k,l}(\alpha_s). \end{aligned} \tag{19}$$

Recall that because

$$W_{k,l}(p)|_{p\to\infty} = \frac{t_0}{p} \delta_{k,0}\delta_{l,0} + O(1/p^2), \tag{20}$$

all $W_{k,l}(p)$ are total derivatives,

$$W_{k,l}(p) = \frac{\partial}{\partial V(p)} \mathcal{F}_{k,l}, \quad k, l \geq 0. \tag{21}$$

The planar limit solution for $W_{0,0}(p)$ exactly coincides with the one in the one-matrix model. The normalizing conditions seem to be those ensuring the coincidence of chemical potentials in all orders of the perturbation theory

$$\oint_{B_i} W_{k,l}(\xi, p_2, \ldots, p_s) = 0, \quad \text{for} \quad k + l > 0. \tag{22}$$

Note that in the Seiberg–Witten theory, these conditions must be merely replaced by the vanishing conditions for the A-cycle integrals.

The formal solution to (19) is given by the integral [5]:

$$W_{k,l}(p) = \oint_{\mathcal{C}_D^{(q)}} dq\, dE_{q,\bar{q}}(p) \frac{1}{2y(q)} \Big(\sum_{\substack{0\le k_1\le k, 0\le l_1\le l\\ 0<k_1+l_1<k+l}} W_{k_1,l_1}(p) W_{k-k_1,l-l_1}(p) + \frac{\partial}{\partial V(p)} W_{k-1,l}(p) + \frac{\partial}{\partial p} W_{k,l-1}(p) \Big). \tag{23}$$

Here the contour $\mathcal{C}_D^{(q)}$ lies on the physical leaf, encircles all cuts of the function $y(q)$, and leaves outside all the other possible singularities (poles and zeros). In the next section we develop the diagrammatic representation for terms of these recurrent relations.

3 Graphical representation. Spatial derivative.

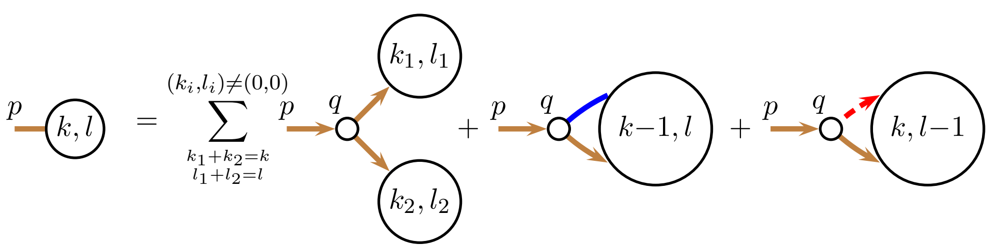
The diagrammatic technique of [5] was based on the diagrammatic technique developed in [4] for the Hermitian one-matrix model, but contained new elements. The original technique of [4] contained two types of the propagators: arrowed lines corresponding to $dE_{q,\bar{q}}(p)$ and nonarrowed lines corresponding to $B(p,q)$. Simultaneously, the two-point correlation function

$$\frac{\partial}{\partial V(p)} W_{0,0}(q) = B(p,q) - \frac{1}{(p-q)^2},$$

and the action of the loop insertion operator on it produces the three-point correlation function described by the Rauch variation formulas that result in the appearance of the three-point vertex denoted by light circle and containing the integration over the contour $\mathcal{C}_D^{(q)}$ as in (23) with the weight $1/(2y(q))$ First of all, the recursion kernel in (23) coincides with the recursion kernel in [4], but to comply with the new term, $\zeta \frac{\partial}{\partial p} W_{k,l-1}(p)$, appearing in the right-hand side, we must introduce a *new* propagator $dpdy(q)$ denoted by the dashed line.

The graphical representation for a solution of loop equation (17) then looks as follows. The multipoint resolvent $W_{k,l}(p_1,\ldots,p_s)$ is represented by the block with s external legs and with the double index k,l. Among these legs one, say p_1, is selected to be the root of maximal tree subgraph composed from arrowed propagators and establishing a partial ordering on the set of vertices of a graph.

We present the derivative $\frac{\partial}{\partial p_1} W_{k,l-1}(p_1,\ldots,p_s)$ as the block with $s+1$ external legs, one of which is the dashed leg that starts also at the vertex p_1. For instance, for the one-point resolvent expansion term $W_{k,l}(p)$, we obtain [5]:



(24)

In the case of multipoint resolvent $W_{k,l}(p_1,\ldots,p_s)$ the picture is similar, we replace p by p_1 and add $s-1$ remaining external legs to the diagrams in the right-hand side; then, in the first diagram we must distribute the added $s-1$ external legs over two subdiagrams in an arbitrary way and take a sum over all possible insertions that leave the subdiagrams $W_{k_i,l_i}(q,p_{j_1},\ldots,p_{j_{s_i}})$ "stable." Recall that $W_{k,l}(p_1,\ldots,p_s)$ is called *stable* (for a nonempty set of $p_1,\ldots,p_s$) if either $k>0$ or $l>0$ or $s>2$.

We can then in turn present the term $W_{k,l-1}(q,p_2,\ldots,p_s)$ in the form (23) with the recursion kernel $dE_{\eta,\bar{\eta}}(q)/(2y(\eta))$ integrated with some function $F(\eta)$, the rest of the diagram, which we do not

specify here. Let us consider the action of the spatial derivative $\partial/\partial q$ on $W_{k,l-1}(q,p_2,\dots,p_s)$. We place the *starting point* of the dashed directed edge at the point q and associate just dx with this starting point. The first object (on which the derivative actually acts) is $dE_{\eta,\bar\eta}(q)$, then comes the vertex with the integration $\frac{1}{2\pi i}\oint_{\mathcal{C}^{(\eta)}_{\mathcal{D}}}\frac{d\eta}{y(\eta)}$, then the function $F(\eta)$. We can present the action of the derivative via the contour integral around q with the kernel $B(q,\xi)$:

$$\frac{\partial}{\partial q}\left(\oint_{\mathcal{C}^{(\eta)}_{\mathcal{D}}}\frac{dE_{\eta,\bar\eta}(q)d\eta}{2\pi i\ y(\eta)}F(\eta)\right)=\operatorname*{Res}_{\xi\to q}\oint_{\mathcal{C}^{(\eta)}_{\mathcal{D}}}\frac{B(q,\xi)dE_{\eta,\bar\eta}(\xi)}{d\xi}\frac{d\eta}{2\pi i\ y(\eta)}F(\eta), \tag{25}$$

where q lies outside the integration contour for η. The integral over ξ is nonsingular at infinity, so we can deform the integration contour from $\mathcal{C}_q$ to $\mathcal{C}^{(\xi)}_{\mathcal{D}}>\mathcal{C}^{(\eta)}_{\mathcal{D}}$,[1]

$$\oint_{\mathcal{C}_q}\frac{B(q,\xi)dE_{\eta,\bar\eta}(\xi)}{2\pi i\ d\xi}\oint_{\mathcal{C}^{(\eta)}_{\mathcal{D}}}\frac{d\eta}{2\pi i\ y(\eta)}F(\eta)=-\oint_{\mathcal{C}^{(\xi)}_{\mathcal{D}}>\mathcal{C}^{(\eta)}_{\mathcal{D}}}\frac{B(p_1,\xi)dE_{\eta,\bar\eta}(\xi)}{2\pi i\ d\xi}\frac{d\eta}{2\pi i\ y(\eta)}F(\eta) \tag{26}$$

We now push the contour for ξ through the contour for η, picking residues at the poles in ξ, at $\xi=\eta$, $\xi=\bar\eta$ and at the branch points. We then obtain

$$-\oint_{\mathcal{C}^{(\eta)}_{\mathcal{D}}}\sum_\alpha\operatorname*{Res}_{\xi\to\mu_\alpha}\frac{B(q,\xi)dE_{\eta,\bar\eta}(\xi)}{d\xi}\frac{d\eta}{2\pi i\ y(\eta)}F(\eta)-\oint_{\mathcal{C}^{(\eta)}_{\mathcal{D}}}\frac{B(q,\eta)}{2\pi i\ y(\eta)}F(\eta)+\oint_{\mathcal{C}^{(\eta)}_{\mathcal{D}}}\frac{B(q,\bar\eta)}{2\pi i\ y(\eta)}F(\eta), \tag{27}$$

where the first integral has only simple poles at the branch points. The main trick stems from that the residue remains unchanged if we apply the l'Hôpital rule and replace $B(q,\xi)/dy(\xi)$ by $dE_{\xi,\bar\xi}(q)/2y(\xi)$, thus replacing $B(q,\xi)$ by $dE_{\xi,\bar\xi}(q)dy(\xi)/2y(\xi)$[2]; we then obtain in the right-hand side of (26):

$$-\oint_{\mathcal{C}^{(\eta)}_{\mathcal{D}}}\sum_\alpha\operatorname*{Res}_{\xi\to\mu_\alpha}\frac{dE_{\xi,\bar\xi}(p_1)\,dy(\xi)\,dE_{\eta,\bar\eta}(\xi)}{2y(\xi)\,d\xi}\frac{d\eta}{2\pi i\ y(\eta)}F(\eta)-\oint_{\mathcal{C}^{(\eta)}_{\mathcal{D}}}\frac{B(q,\eta)-B(q,\bar\eta)}{2\pi i\ y(\eta)}F(\eta).$$

Pushing contour of integration for ξ around the branch points back through the contour for η, we pick residues at $\xi=\eta$ and $\xi=\bar\eta$, which both give the same contribution, and obtain

$$\begin{aligned}&-\oint_{\mathcal{C}^{(\xi)}_{\mathcal{D}}>\mathcal{C}^{(\eta)}_{\mathcal{D}}}\frac{dE_{\xi,\bar\xi}(q)\,dy(\xi)\,dE_{\eta,\bar\eta}(\xi)}{2\pi i\ 2y(\xi)\,d\xi}\frac{d\eta}{2\pi i\ y(\eta)}F(\eta)\\&+\oint_{\mathcal{C}^{(\eta)}_{\mathcal{D}}}\frac{dE_{\eta,\bar\eta}(q)\,dy(\eta)}{y(\eta)\,d\eta}\frac{d\eta}{2\pi i\ y(\eta)}F(\eta)-\oint_{\mathcal{C}^{(\eta)}_{\mathcal{D}}}\frac{B(q,\eta)-B(q,\bar\eta)}{2\pi i\ y(\eta)}F(\eta).\end{aligned} \tag{28}$$

We evaluate the last two terms by parts therefore obtaining

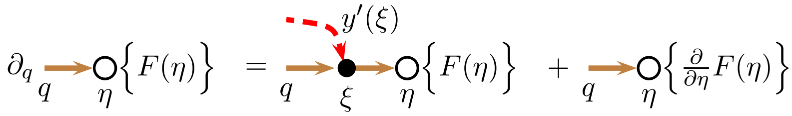
$$-\oint_{\mathcal{C}^{(\xi)}_{\mathcal{D}}>\mathcal{C}^{(\eta)}_{\mathcal{D}}}\frac{dE_{\xi,\bar\xi}(p_1)\,dy(\xi)\,dE_{\eta,\bar\eta}(\xi)}{2\pi i\ 2y(\xi)\,d\xi}\frac{d\eta}{2\pi i\ y(\eta)}F(\eta)+\oint_{\mathcal{C}^{(\eta)}_{\mathcal{D}}}\frac{dE_{\eta,\bar\eta}(q)\,d\eta}{2\pi i\ y(\eta)}\frac{\partial F(\eta)}{\partial\eta}. \tag{29}$$

We have thus found that

$$\begin{aligned}&\frac{\partial}{\partial_q}\left(\oint_{\mathcal{C}^{(\eta)}_{\mathcal{D}}}\frac{dE_{\eta,\bar\eta}(q)d\eta}{2\pi i\ y(\eta)}F(\eta)\right)\\=&\ -\oint_{\mathcal{C}^{(\xi)}_{\mathcal{D}}>\mathcal{C}^{(\eta)}_{\mathcal{D}}}\frac{dE_{\xi,\bar\xi}(q)\,y'(\xi)\,dE_{\eta,\bar\eta}(\xi)}{2\pi i\ 2y(\xi)}\frac{d\eta}{2\pi i\ y(\eta)}F(\eta)+\oint_{\mathcal{C}^{(\eta)}_{\mathcal{D}}}\frac{dE_{\eta,\bar\eta}(p_1)\,d\eta}{2\pi i\ y(\eta)}\frac{\partial F(\eta)}{\partial\eta},\end{aligned} \tag{30}$$

[1]Here and hereafter, the comparison of contours indicates their ordering.
[2]We therefore "imitate" pole terms by inserting dy/y.

and we can graphically present the action of the derivative as follows:



(31)

In these relations and in what follows, we use light circles ("white" vertices) to denote the integrations with the weight $1/(2y(\eta))$ and "black" vertices to indicate integrations with weights containing the first or higher derivatives $y^{(s)}(\eta)$ in the numerator.

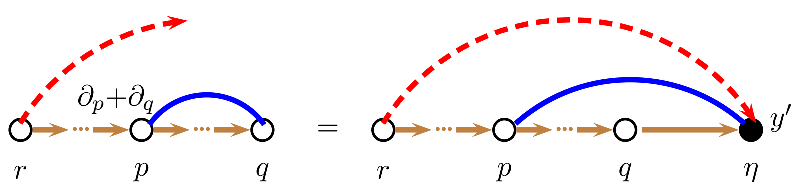
We also assume the ordering from left to right in all the diagrams below.

We therefore see that using relation (31) we can push the differentiation along the arrowed propagators of a graph. It remains to determine the action of the derivative on internal nonarrowed propagators. But for these propagators (since it has two ends), having the term with derivative from the one side, we necessarily come also to the term with the derivative from the other side *if we act by the dashed line that was commenced before the outer end of this propagator*. Combining these terms, we obtain

$$\partial_p B(p,q) + \partial_q B(p,q) = \oint_{\mathcal{C}_p \cup \mathcal{C}_q} \frac{B(p,\xi)B(\xi,q)}{2\pi i\ d\xi},$$

and we can deform this contour to the sum of contours only around the branching points (sum of residues). Then we can again introduce $y'(\xi)d\xi/y(\xi)$ and integrate out one of the Bergman kernels (the one that is adjacent to the point q if $p > q$ or p if $q > p$; recall that, by condition, the points p and q must be comparable).

That is, we have



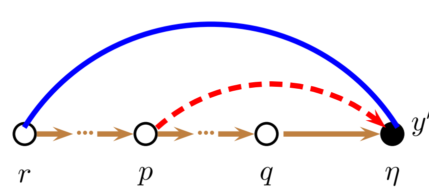
(32)

We must therefore improve the diagrammatic technique of β-model in comparison with the Hermitian one-matrix model by including the dashed lines. They are not actual propagators, but these lines ensure the proper combinatorics of diagrams. Indeed, from (31) it follows that the derivative action on the "beginning" of the dashed line is null, $\partial_p dp = 0$, and when this derivative acts on the "end" of this line with the black vertex, we merely have

$$\partial_q y'(q) = y''(q), \tag{33}$$

which we denote symbolically as *two* dashed propagators ending at the same black vertex. If we continue to act by derivatives $\partial/\partial q$, then, obviously, when k dashed propagators are terminated at the *same* vertex, we have the kth order derivative $y^{(k)}(q)$ corresponding to them.

Another lesson from studying the action of the spatial derivative is that *not all* diagrams are possible: for example, the diagram containing the vertex



never appear in our approach. Later on we formulate all the selection rules for the diagrams as well as present all types of possible vertices.

We also introduce the black-and-white coloring of vertices in accordance with the rule: if there is no factors $y^{(k)}(q)$ standing by the vertex, it is white; if there are such factors, the vertex is painted black.

We need now to produce the action of the loop insertion operator in the new setting. We calculate the action of $\partial/\partial V(r)$ on the dashed propagator. Obviously,

$$\frac{\partial}{\partial V(r)} y^{(k)}(q) = \frac{\partial^k}{\partial q^k} B(r,q), \tag{34}$$

and any attempt to simplify this expression or to reduce it to a combination of previously introduced diagrammatic elements fails. This means that we must consider it a *new* element of the diagrammatic technique. We indicate it by preserving k dashed arrows still terminating at the vertex q with the added nonarrowed solid line corresponding to the new propagator $B(r,q)$. The vertex then change the coloring from black to white because it contains $y^{(k)}(q)$ factors no more and we therefore assume that these derivatives act on $B(r,q)$.

An accurate analysis demonstrates nevertheless that we must include also vertices with the mixed action of spatial derivatives: on $B(r,q)$ and on $y^{(s)}(q)$ standing at this vertex (which is then also painted black). These vertices were also missed in the original paper [5].

We leave the detailed proof for subsequent publications and present in the next section the *complete set of rules* for constructing diagrammatic representation for $W_{k,l}(p_1,\dots,p_s$ for all k and l.

4 Feynman diagrammatic rules

We now describe the diagrammatic technique that results from the action of ∂_q—the spatial derivative and the action of the loop insertion operator $\partial/\partial V(q)$ with describing new vertices and selection rules.

In this diagrammatic technique, all the solid arrowed propagators $dE_{q,\bar q}(p)$ are free of spatial derivatives, vertices always contain the factor $1/(2y(q))$ and may or may not contain additional derivatives $y^{(s)}(q)$ in the numerator and the solid nonarrowed propagators corresponding to B-kernels may contain additional spatial derivatives acting only on the innermost end of the propagator (recall that B-propagators can join *only* vertices comparable in the sense of partial ordering established by the tree of arrowed propagators.

In the graphical representation below the ordering of vertices is implied from left to right.

The diagrammatic technique for constructing resolvents $W_{k,l}(p_1,\dots,p_s)$ is as follows.

We take the sum over connected graphs containing k internal nonarrowed propagators (Bergmann kernels), l dashed arrowed propagators (those are always internal "derivatives") and s external legs such that

- we begin constructing each graph by constructing the maximum rooted directed tree subgraph composed from the propagators $dE_{q,\bar q}(p)$ (q here is the inner vertex with respect to p, the arrow is pointed toward q) and take the sum over all choices of such rooted tree subgraphs;
- we choose the external vertex p_1 to be the root of the tree, one and the same p_1 for all the graphs;
- all other external legs are nonarrowed propagators B; we call propagators dE and B solid as we set solid lines corresponding to them; all the external points of the graph are considered outermost points;

- other internal edges (i.e., edges that are incident only to the graph vertices and not to $p_1, \dots, p_s$) are either B (there are exactly k of them) or dashed arrowed edges (there are exactly l of them);

- rooted tree subgraph establishes a partial ordering on the set of internal vertices; *we allow internal B edges and dashed edges to connect only comparable vertices.* Dashed arrows are directed always inward, but can begin and terminate at the same vertex; if such a vertex exist, it must be an innermost vertex of the graph. The B-lines can also begin and terminate at the same vertex; in this case, this vertex is also necessarily an innermost vertex of the graph.

- A partial ordering establishes the order of taking residues: we start from innermost vertices and continues toward the root of the tree.

- All s external legs p_j are outside all the integrations.

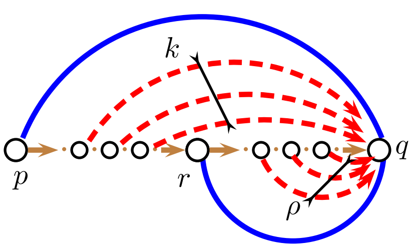
- All the internal vertices are incident to exactly one incoming arrowed line and can be incident up to two other ends of solid lines.

- The vertices of the graph are painted either black or white (depending on whether they respectively contain or do not contain the factors $y^{(k)}(q)$) and must satisfy the *selection rules* below.

- All the graphs satisfying all the above properties enter the final answer with the standard symmetry coefficients that are the reciprocal volumes of the discrete automorphism groups of the graphs.

The selection rules follow from the complete *list of ten types of vertices* below. Their origin will be partly clarified in the next section when considering the action of the loop insertion operator on the elements of the diagrammatic technique. The general proof will be published elsewhere.

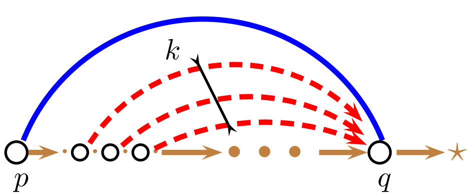
The vertices with three adjacent solid lines



$$\sim \oint_{\mathcal{C}_{\mathcal{D}}^{(q)}} dE_{q,\bar q}(\bullet) \frac{dq}{2\pi i\ y(q)} \frac{\partial^\rho}{\partial q^\rho}\Big(B(r,q)\frac{\partial^k}{\partial q^k}(B(p,q))\Big), \qquad (35)$$

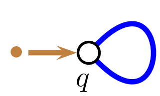
$$k \geq 0,\ \ \rho \geq 0$$

Here the vertex p must be outside all the starting points of dashed lines terminating at q and can be an external vertex. The vertex r can be external if $k = 0$.

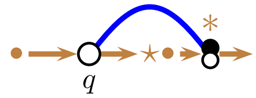


$$\sim \oint_{\mathcal{C}_{\mathcal{D}}^{(q)}} dE_{q,\bar q}(\bullet) \frac{dq}{2\pi i\ y(q)} \frac{\partial^k}{\partial q^k}(B(p,q)) dE_{\star,\bar\star}(q),\ \ k \geq 0. \qquad (36)$$

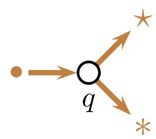
Here the vertex p must be outside all the starting points of dashed lines acting on the vertex q and can be external.



$$\sim \oint_{\mathcal{C}_{\mathcal{D}}^{(q)}} dE_{q,\bar q}(\bullet) \frac{dq}{2\pi i\ y(q)} B(q,\bar q) \qquad (37)$$

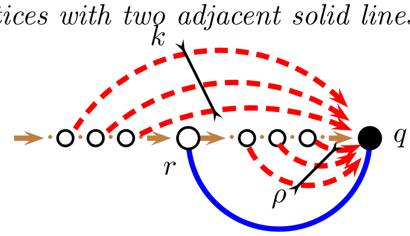


$$\sim \oint_{\mathcal{C}_{\mathcal{D}}^{(q)}} dE_{q,\bar{q}}(\bullet)\frac{dq}{2\pi i\ y(q)}B(q,*)dE_{\star,\bar{\star}}(q) \tag{38}$$



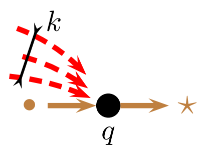
$$\sim \oint_{\mathcal{C}_{\mathcal{D}}^{(q)}} dE_{q,\bar{q}}(\bullet)\frac{dq}{2\pi i\ y(q)}dE_{*,\bar{*}}(q)dE_{\star,\bar{\star}}(q) \tag{39}$$

The vertices with two adjacent solid lines

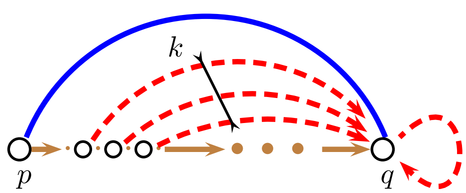


$$\sim \oint_{\mathcal{C}_{\mathcal{D}}^{(q)}} dE_{q,\bar{q}}(\bullet)\frac{dq}{2\pi i\ y(q)}\frac{\partial^\rho}{\partial q^\rho}\Big(B(r,q)y^{(k)}(q)\Big),$$
$$k>0,\ \ \rho\geq 0. \tag{40}$$

Here, note that k *must be greater than zero*, so the point r cannot be external,

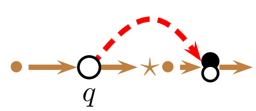


$$\sim \oint_{\mathcal{C}_{\mathcal{D}}^{(q)}} dE_{q,\bar{q}}(\bullet)\frac{dq}{2\pi i\ y(q)}y^{(k)}(q)dE_{\star,\bar{\star}}(q),\ \ k>0. \tag{41}$$



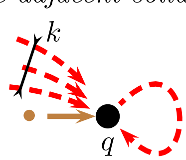
$$\sim \oint_{\mathcal{C}_{\mathcal{D}}^{(q)}} dE_{q,\bar{q}}(\bullet)\frac{dq}{2\pi i\ y(q)}\frac{\partial^{k+1}}{\partial q^{k+1}}B(p,q),\ \ k\geq 0. \tag{42}$$

Here the vertex p must be outside all the starting points of dashed lines acting on the vertex q and can be external.



$$\sim \oint_{\mathcal{C}_{\mathcal{D}}^{(q)}} dE_{q,\bar{q}}(\bullet)\frac{dq}{2\pi i\ y(q)}dE_{\star,\bar{\star}}(q) \tag{43}$$

The vertex with one adjacent solid line



$$\sim \oint_{\mathcal{C}_{\mathcal{D}}^{(q)}} dE_{q,\bar{q}}(\bullet)\frac{dq}{2\pi i\ y(q)}y^{(k+1)}(q),\ \ k\geq 0. \tag{44}$$

When calculating $W_{k,l}(p_1,\ldots,p_s)$ the order of integration contours (previously the order of taking residues at the branching points in the case of the Hermitian one- and two-matrix models) is prescribed by the ordering of vertices in the tree subgraph: the closer is the vertex to the root, the more outer is the integration contour. In contrast to the Hermitian matrix model case, the integration cannot be reduced to taking residues at the branch points only; all internal integrations can be nevertheless reduced to sums of residues, but these sums may now include residues at zeros of the additional polynomial $M(p)$ on the nonphysical sheet and, possibly, at the point ∞_-.

As in the Hermitian matrix-model case, we use the H-operator constructed below to invert the action of the loop insertion operator and obtain the expression for the free energy itself.

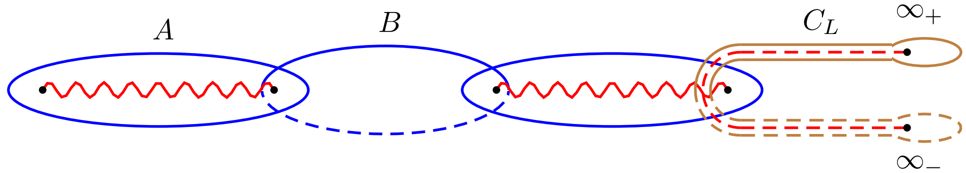


Figure 1: Cuts in the λ-, or "eigenvalue," plane for the planar limit of the β-model (coinciding with the planar limit of the Hermitian one-matrix model). The eigenvalues are supposed to be located "on" the cuts (curly lines). We add the logarithmic cut between two copies of the infinity on two sheets of the hyperelliptic Riemann surface in order to calculate the derivative w.r.t. the variable t_0; C_L is the corresponding regularized integration contour.

5 Inverting the loop insertion operator. Free energy

The H-operator. We now introduce the operator (the functional on 1-forms) that is inverse to loop insertion operator (8). For a form $df(x)$, let[3]

$$H_x df(x) = \frac{1}{2}\mathrm{res}_{\infty_+} V(x)df(x) - \frac{1}{2}\mathrm{res}_{\infty_-} V(x)df(x) - t_0 \int_{\infty_-}^{\infty_+} df(x) - \sum_{i=1}^{n-1} S_i \oint_{B_i} df(x). \tag{45}$$

The arrangement of the integration contours is as in Fig. 1.

The H-operator, in contrast to the loop insertion operator, therefore reduce by one the degree of the (symmetrized) form associated with an s-point correlation function

$$H : \Omega^s_{1,0} \mapsto \Omega^{s-1}_{1,0}. \tag{46}$$

We calculate the action of H on the Bergmann bidifferential $B(x,q)$ using again the Riemann bilinear identities. We first note that as $B(x,q) = \partial_x dE_{x,q_0}(q)$, we can evaluate residues at infinities by parts. Then, since $dE_{x,q_0}(q)$ is regular at infinities, we substitute $2y(x) + 2t_0/x$ for $V'(x)$ as $x \to \infty_+$ and $-2y(x) + 2t_0/x$ for $V'(x)$ as $x \to \infty_-$ thus obtaining

$$\begin{aligned} -\mathrm{res}_{\infty_+} \left(y(x) + \frac{t_0}{x}\right) dE_{x,q_0}(q)dx + \mathrm{res}_{\infty_-} \left(-y(x) + \frac{t_0}{x}\right) dE_{x,q_0}(q)dx \\ -t_0 dE_{x,q_0}(q)\Big|_{x=\infty_-}^{x=\infty_+} - \sum_{i=1}^{n-1} S_i \oint_{B_i} B(q,x). \end{aligned} \tag{47}$$

Whereas the cancelation of terms containing t_0 is obvious, it remains only to take the combination of residues at infinities involving $y(x)$. For this, we cut the surface along A- and B-cycles taking into account the residue at $x = q$. The boundary integrals on two sides of the cut at B_i then differ by $dE_{x,q_0}(q) - dE_{x+\oint_{A_i},q_0}(q) = 0$, while the integrals on the two sides of the cut at A_i differ by $dE_{x,q_0}(q) - dE_{x+\oint_{B_i},q_0}(q) = \oint_{B_i} B(q,x)$, and the boundary term therefore becomes

$$\sum_{i=1}^{n-1} \oint_{A_i} y(x)dx \oint_{B_i} B(q,\xi),$$

which exactly cancels the last term in (47). Only the contribution from the pole at $x = q$ then survives, and this contribution is just $-y(q)$. We have therefore proved that

$$H_x \cdot B(x,q) = -y(q)dq. \tag{48}$$

[3]This definition works well when acting on 1-forms regular at infinities. Otherwise (say, in the case of $W_0(p)$), the integral in the third term must be regularized by replacing it by the integral along the contour C_L depicted in Fig. 1.

Let us now consider the action of the operator H_x given by (45) on $W_{k,l}(x)$ subsequently evaluating the action of loop insertion operator on the result. Due to the commutation relations between H_x and $\partial/\partial V(x)$,

$$H \circ \partial/\partial V - \partial/\partial V \circ H = \mathrm{Id} \quad \text{modulo the kernels,}$$

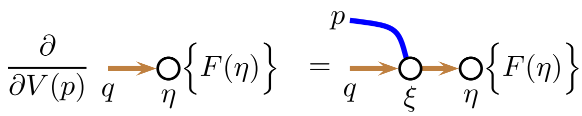
we have,

$$\frac{\partial}{\partial V(p)}\left(H_x \cdot W_{k,l}(x)\right) = W_{k,l}(p) + H_x \cdot W_{k,l}(x,p). \tag{49}$$

For the second term, due to the symmetry of $W_{k,l}(p,q)$, we may choose the point p to be the root of the tree subgraphs. Then, the operator H_x always acts on $B(x,\xi)$ (or, possibly, on its derivatives w.r.t. ξ) where ξ are some integration variables of internal vertices.

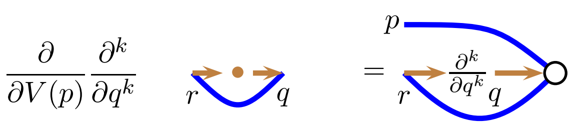
Let us consider the action of $\partial/\partial V(p)$ on the elements of the Feynman diagram technique in Sec. 4. Here we have three different cases; in all of them the variable p is external.

- When acting on the arrowed propagator followed by a (white or black) vertex, we use the standard variational relation in (50) below, where, again, $F(\eta)$ is any combination of elements of the diagrammatic technique,



(50)

- When acting on nonarrowed internal propagator $\frac{\partial^k}{\partial q^k}B(r,q)$, $r > q$, $k \geq 0$, we apply the relation



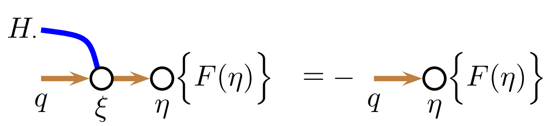
(51)

  without subsequent expanding the action of the derivative $\frac{\partial^k}{\partial q^k}$ into a sum of diagrams.

- Eventually, when acting on the factors $y^{(k)}(q)$ standing at the black vertices, using relation (34) we obtain from the diagrams (40), (41), and (44) the respective diagrams (35), (36), and (42).

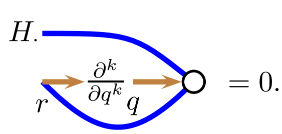
We now consider the *inverse* action of the H-operator in all three cases.

In the first case where it exists an outgoing arrowed propagator $dE_{\eta,\bar{\eta}}(\xi)$ we can push the integration contour for ξ through the one for p; the only contribution comes from the pole at $\xi = p$ (with the *opposite* sign due to the choice of contour directions in Fig. 1). Graphically, we have



(52)

In the second case, the vertex ξ in (51) is an innermost vertex (i.e., there is no arrowed edges coming out of it). The 1-form $y(\xi)d\xi$ arising under the action of H (48) cancels the corresponding form in the integration expression, the expression becomes regular at the branching point and the residue vanishes. Graphically, we have for $k \geq 0$,



(53)

Eventually, in the third case, the action of the H-operator just erases the new B-propagator with the external variable p and restore the term $y^{(k)}(q)$, and from the diagrams (35), (36), and (42) we return to the respective initial diagrams (40), (41), and (44) changing back the color of vertices from white to black. Note that in this case we have the plus sign upon the action of the operator H.

For $H_x \cdot W_{k,l}(x,p) = H_x \cdot \frac{\partial}{\partial V(x)} W_{k,l}(p)$, we obtain that for each solid arrowed edge, on which the action of $\partial/\partial V(x)$ produces the new (white) vertex, the inverse action of H_x gives the factor -1, on each nonarrowed edge, on which the action of $\partial/\partial V(x)$ produces the new vertex accordingly to (51), the inverse action of H_x just gives zero, and at each black vertex, at which the action of the loop insertion operator changes the color to white and adds a new B-propagator, the inverse action of H_x gives the factor $+1$.

As the total number of arrowed edges coincides with the total number of vertices and the contributions of black vertices are opposite to the contributions of arrowed edges, the total factor on which the diagram is multiplied is minus the number of white vertices, which is exactly $2k + l - 1$ for *any* graph contributing to $W_{k,l}(p)$. (For the s-point correlation function $W_{k,l}(p_1, \ldots, p_s)$ this number is $2k + l + s - 2$.) We then have

$$H_x \cdot W_{k,l}(x,p) = -(2k + l - 1) W_k(p)$$

and, combining with (49), we obtain

$$\frac{\partial}{\partial V(p)} \left(H_x \cdot W_{k,l}(x) \right) = -(2 - 2k - l) \frac{\partial}{\partial V(p)} \mathcal{F}_{k,l}. \tag{54}$$

We therefore obtain the final answer for the *free energy*:

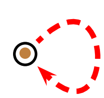
$$\mathcal{F}_{k,l} = \frac{1}{2k + l - 2} H_x \cdot W_{k,l}(x), \tag{55}$$

which enables one to calculate *all* the terms $\mathcal{F}_{k,l}$ except the contribution at $k = 1, l = 0$ (which is the torus approximation in the Hermitian one-matrix model calculated in [3]) and the second-order correction in ζ (the term $\mathcal{F}_{0,2}$). The term $\mathcal{F}_{0,2}$ was calculated in [5], in the next subsection we present the answer.

To calculate *all other* terms of the free energy expansion we only need to introduce the **new vertex** $\odot$ at which we place the (nonlocal) integral term $\oint_{\mathcal{C}^{(\xi)}_{\mathcal{D}}} \frac{d\xi}{2\pi i} \frac{\int_{\bar\xi}^{\xi} y(s)ds}{2y(\xi)}$. *To obtain $\mathcal{F}_{g,l}$ in the above diagrammatic technique we merely remove the root of the tree and replace the first internal vertex (which is always white) by this new (two-valent) vertex, as shown in examples below.*

In the Hermitian matrix model case, it was possible to perform the integration for $\int_{\bar\xi}^{\xi} 2y(s)ds$ through the branch point μ_α in the vicinity of the αth branch point; here it is no more the case and we must consider global integrations. Note, however, that we must introduce nonlocal terms *only* for the very last, outermost, integration; all internal integrations can be performed by taking residues at branch points and at the zeros of the polynomial $M(p)$.

For example, the first correction to the free energy is

$$-1 \cdot \mathcal{F}_{0,1} =$$


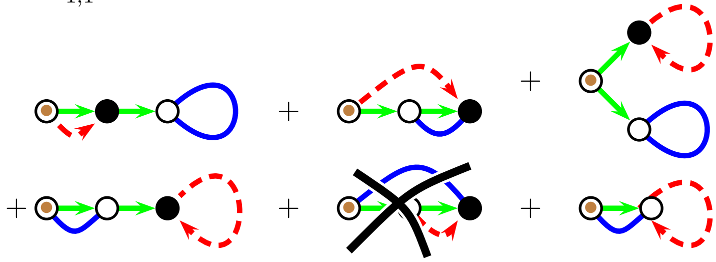
(56)

By integration by parts (in which we make substitution at q and $\bar q$, which cancels the factor 2 in the denominator), we obtain

$$\mathcal{F}_{0,1} = -\oint_{\mathcal{C}^{(q)}_{\mathcal{D}}} \frac{\int_{\bar q}^{q} y}{2y(q)} y'(q) \frac{dq}{2\pi i} = \oint_{\mathcal{C}^{(q)}_{\mathcal{D}}} y(q) \log y(q) \frac{dq}{2\pi i},$$

i.e., we come to the semiclassical Dyson term (the entropy). Other lower-order terms of the free-energy expansion may also have a physical interpretation, as we demonstrate on the example of the term $\mathcal{F}_{0,2}$.

For the term $\mathcal{F}_{1,1}$ we have

$\mathcal{F}_{1,1} =$ 

where we explicitly crossed out the diagram that is forbidden by the selection rules. So, the actual answer comprises five admitted diagrams.

5.1 Calculating $\mathcal{F}_{0,2}$

It was shown in [5] that the diagrammatic representation formally applied to $\mathcal{F}_{0,2}$ just gives zero.

We can however make a guess for the actual $\mathcal{F}_{0,2}$. It was proposed in [5] to interpret it as the Polyakov's gravitational anomaly term $\iint R\frac{1}{\Delta}R$, where R is the curvature (of two-dimensional metric) and $1/\Delta$ is the Green's function for the Laplace operator, which in our case is the logarithm of the Prime form. The curvature is expressed through the function y as $R \sim y'/y$. That is, we have two natural candidates for $\mathcal{F}_{0,2}$:

$$\mathcal{F}_{0,2} \sim \iint dq\,dp \frac{y'(q)}{y(q)} \log E(p,q) \frac{y'(p)}{y(p)},$$

where E is the Prime form, or

$$\mathcal{F}_{0,2} \sim \iint dq\,dp \log y(q) B(p,q) \log y(p),$$

but neither of these expressions is well defined. The first one develops the logarithmic cut at $p = q$ and cannot be written in a contour-independent way; moreover, both these expressions are divergent when integrating along the support. We therefore must find another representation imitating this term. A proper choice turns out to be the one in which we integrate by part only once therefore producing the expression of the form

$$\oint_{\mathcal{C}_{\mathcal{D}}^{(q)}} \frac{dq}{2\pi i} \frac{y'(q)}{y(q)} \int_{\mathcal{D}} dE_{q,\bar{q}}(p) \log y(p) dp, \tag{57}$$

where the second integral is taken exactly along the eigenvalue support $\mathcal{D}$.

Variation of the logarithmic term in (57) can be presented already in the form of the contour integral. Referring the reader for details to [5], we present here only the result.

Introducing $M_\alpha^{(1)} \equiv Q(p)|_{p=\mu_\alpha}$ to be the first *moments* of the total potential V and $\Delta(\mu)$ to be the Vandermonde determinant of the branch points μ_α, we have

$$\mathcal{F}_{0,2} = -\oint_{\mathcal{C}_{\mathcal{D}}^{(q)}} \frac{dq}{2\pi i} \frac{y'(q)}{y(q)} \int_{D} dE_{q,\bar{q}}(p) \log y(p) dp - \frac{1}{3} \log \left(\prod_{\alpha=1}^{2n} M_\alpha^{(1)} \Delta(\mu) \right), \tag{58}$$

and we indeed have *quantum correction* term (the second term in (58)).

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